§ Properties of arc length

<u>Definition</u>: Let $\alpha : I \to \mathbb{R}^3$ be a curve, and $[a,b] \subset I$.

The length of α from a to b is defined as

$$(*): \quad \boxed{L_a^b(\alpha) := \int_a^b |\alpha'(t)|\, dt}$$

<u>FACT</u>: Length is a geometric quantity.

i.e. it is invariant under rigid motions and reparametrizations.

<u>Definition</u>: A rigid motion $\phi : \mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ is an isometry of $\mathbb{R}^n$ (as a metric space with respect to the Euclidean distance).

$$\text{i.e. } |\phi(p) - \phi(q)| = |p - q| \quad \forall\, p, q \in \mathbb{R}^n$$

It is well-known that any such ϕ is an affine map of the form:

$$\boxed{\phi(x) = Ax + b} \qquad \forall\, x \in \mathbb{R}^n$$

where $A \in O(n)$, i.e. $AA^t = I = A^tA$, and $b \in \mathbb{R}^n$.

Definition: If $\det A > 0$, ϕ is orientation-preserving.

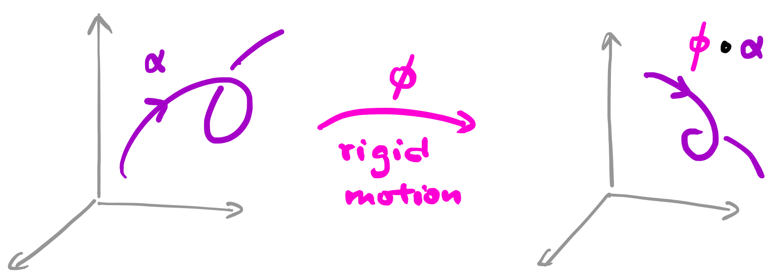
If $\det A < 0$, ϕ is orientation-reversing.

Note: $\det A = \pm 1$ if $A \in O(n)$.

Proposition: Rigid motions preserve length of curves, i.e. if $\alpha : I \to \mathbb{R}^3$ is a curve and $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid motion, then $\phi \circ \alpha : I \to \mathbb{R}^3$ is also a curve and

$$\boxed{L_a^b(\alpha) = L_a^b(\phi \circ \alpha)} \quad \text{for any } [a,b] \subseteq I.$$

Proof: Exercise.



length unchanged!

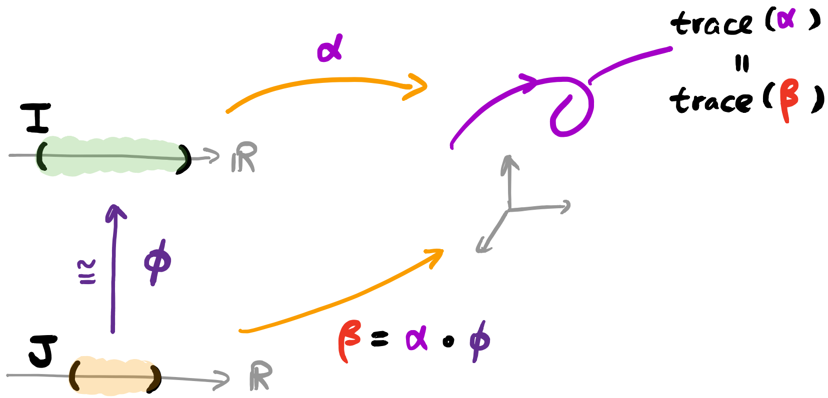
Proposition: Straight lines are the (unique) shortest curves joining two given points in $\mathbb{R}^3$.

Proof: Exercise.

Definition: Let $\alpha : I \to \mathbb{R}^3$ be a curve. For any diffeomorphism $\phi : J \subseteq \mathbb{R} \to I$, one can define a new curve

$$\boxed{\beta = \alpha \circ \phi : J \to \mathbb{R}^3}$$

which is called a reparametrization of α.



Remark: α and β parametrize the "same" curve, i.e. their images are the same.

Proposition: The length of a curve is invariant under reparametrization, i.e.

$$L_c^d(\beta) = L_a^b(\alpha) \quad \text{if } \phi([c,d]) = [a,b]$$

Proof: $L_a^b(\alpha) = \int_a^b |\alpha'(t)|\, dt$

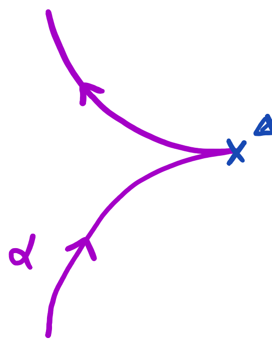
Change of variable formula

$$L_c^d(\beta) = \int_c^d |\beta'(u)|\, du \underset{\text{Rule}}{\overset{\text{Chain}}{=}} \int_c^d |\alpha'(\phi(u))| \underbrace{|\phi'(u)|}_{>0}\, du$$

$\square$

§ Regular Curves

<u>Recall</u>: The trace of a curve may not be "smooth":



NOT "smooth" here!

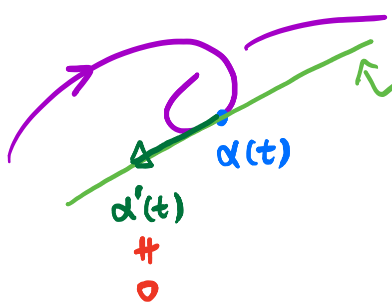
since $\nexists$ good approx. tangent line

at this pt.

<u>Definition</u>: A curve $\alpha : I \to \mathbb{R}^3$ is regular

if $|\alpha'(t)| \neq 0$ for all $t \in I$.

<u>Note</u>: For a regular curve $\alpha : I \to \mathbb{R}^3$,



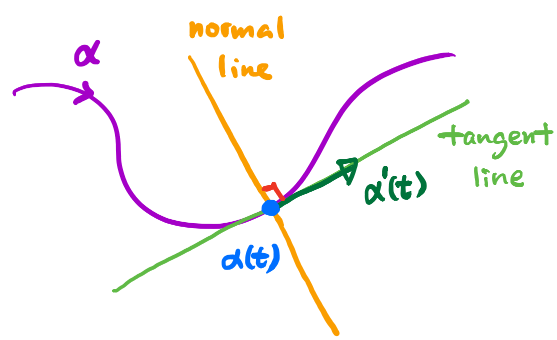
tangent line at time t

is well-defined!

For a regular plane curve

$\alpha : I \to \mathbb{R}^2$

normal line is also defined



Q: What about for space curve?

Q: Is there a "best" way to parametrize a curve?

A: Yes, with constant speed!

Definition: A curve $\alpha : I \to \mathbb{R}^3$ is said to be parametrized by arc length (p.b.a.l.) if

$$\boxed{|\alpha'(t)| = 1 \quad \forall\, t \in I}$$

Remark: If $\alpha : I \to \mathbb{R}^3$ is p.b.a.l., then

$$L_a^b(\alpha) = b - a \quad \text{for any } [a,b] \subset I.$$

Example: Any curve $\alpha : I \to \mathbb{R}^3$ which is p.b.a.l. is regular. ($\because |\alpha'(t)| = 1 \neq 0$)

Theorem: Any regular curve admits a reparametrization by arc length.

(i.e. given a regular curve

$$\alpha : I \to \mathbb{R}^3$$

$\exists$ diffeomorphism $\phi : J \to I$ for some interval $J \subseteq \mathbb{R}$ s.t.

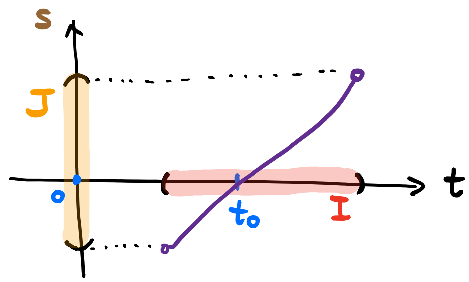
$$\beta = \alpha \circ \phi : J \to \mathbb{R}^3 \text{ is p.b.a.l.})$$

Proof: Fix any $t_0 \in I$. consider the

arc length function from t_0 : $S(t) := \int_{t_0}^{t} |\alpha'(u)|\, du$

Note: S differentiable with $S'(t) = |\alpha'(t)| \overset{\text{regular}}{>} 0 \quad \forall t \in I$

So, S is a continuous, strictly increasing function:



$S : I \xrightarrow{\cong} J \subseteq \mathbb{R}$

has a smooth inverse

$\phi = S^{-1} : J \xrightarrow{\cong} I$.

Define $\beta = \alpha \circ \phi : J \to \mathbb{R}^3$, which is a reparam. of α

Claim: β is p.b.a.l.

Proof of Claim:

$$\beta'(s) \underset{\text{Chain rule}}{=} \alpha'(\phi(s)) \cdot \phi'(s) = \frac{\alpha'(\phi(s))}{S'(\phi(s))} = \underbrace{\frac{\alpha'(\phi(s))}{|\alpha'(\phi(s))|}}_{\text{unit vector!}}$$

$\square$

Example: $\alpha(t) = (2\cos t, 2\sin t)$, $t \in \mathbb{R}$

$$S(t) = \int_0^t |\alpha'(u)|\, du = \int_0^t 2\, du = 2t$$

inverse $\Rightarrow$ $\phi(s) = \frac{s}{2}$, $s \in \mathbb{R}$

$$\beta(s) = \alpha(\phi(s)) = \alpha\left(\frac{s}{2}\right) = \left(2\cos\frac{s}{2}, 2\sin\frac{s}{2}\right), \quad s \in \mathbb{R}$$

Example: Reparametrize the helix by arc length.

$$\alpha(t) = (a\cos t, a\sin t, bt), \quad t \in \mathbb{R}$$

$\alpha'(t) = (-a\sin t, a\cos t, b)$; $|\alpha'(t)| = \sqrt{a^2+b^2}$

$$S(t) = \int_0^t |\alpha'(u)|\, du = \sqrt{a^2+b^2}\, t$$

inverse $t(s) = \dfrac{s}{\sqrt{a^2+b^2}}$

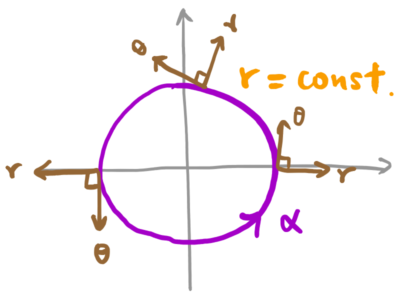
$$\beta(s) = \alpha\left(\frac{s}{\sqrt{a^2+b^2}}\right) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, b\frac{s}{\sqrt{a^2+b^2}}\right)$$

FROM NOW ON, we always assume a curve is p.b.a.l. unless otherwise stated.

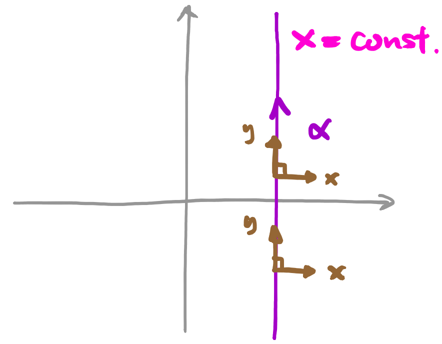
§ Local theory of plane curves

Consider two plane curves:

(r,θ): polar coordinates

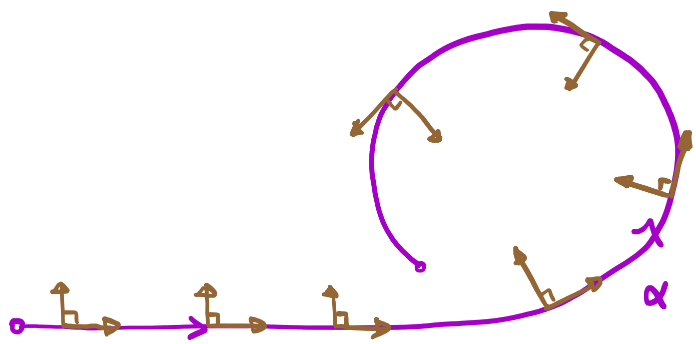


(x,y): rectangular coordinates



Question: What is the "best" coordinate system on a given (regular) curve?

E.g.

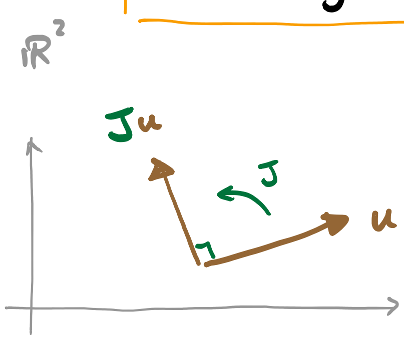


depends on where we are on the curve

$\Downarrow$

"moving frame"

Let $\boxed{J = \text{counterclockwise rotation by } 90^\circ \text{ in } \mathbb{R}^2 \ (\cong \mathbb{C})}$ special in dim. 2



Given any unit vector $u \in \mathbb{R}^2$

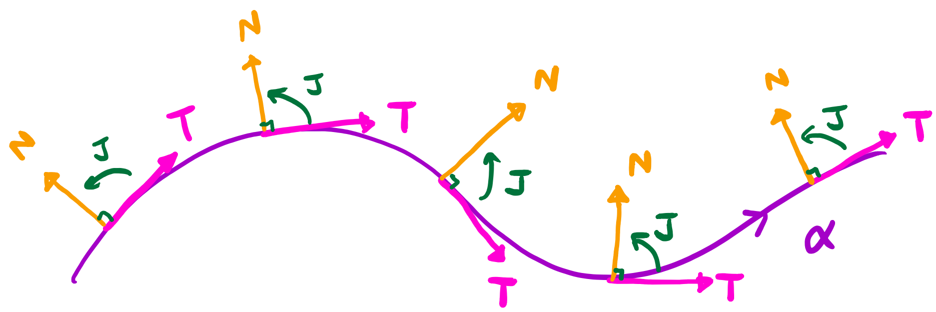
$\{u, Ju\}$ is a (pos. oriented) orthonormal basis (O.N.B.)

called a frame.

Definition: Let $\alpha : I \to \mathbb{R}^2$ be a plane curve p.b.a.l.

Define: $\begin{cases} T(s) = \alpha'(s) & \text{unit tangent} \\ N(s) = J(T(s)) & \text{unit normal} \end{cases}$

$\{T(s), N(s)\}$ Frenet frame along α



Note: $\{T(s), N(s)\}$ O.N.B for each $s \in I$

$$\Rightarrow \begin{cases} \langle T(s), T(s) \rangle \equiv 1 \equiv \langle N(s), N(s) \rangle \quad \cdots\cdots \text{①} \\ \langle T(s), N(s) \rangle \equiv 0 \quad \cdots\cdots\cdots\cdots\cdots \text{②} \end{cases}$$

Differentiate ①:

$$\langle T'(s), T(s) \rangle \equiv 0 \equiv \langle N'(s), N(s) \rangle$$

Differentiate ②: by product rule

$$\langle T'(s), N(s) \rangle + \langle T(s), N'(s) \rangle \equiv 0$$

Hence, $\begin{cases} T'(s) = k(s) N(s) \\ N'(s) = -k(s) T(s) \end{cases}$

Definition: Let $\alpha : I \to \mathbb{R}$ be a curve p.b.a.l. with Frenet frame $\{T(s), N(s)\}$.

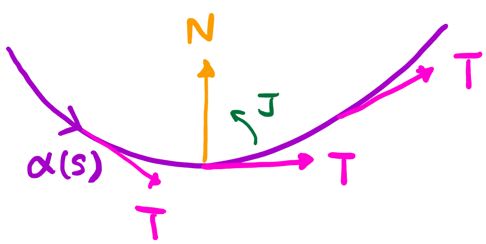
$$\boxed{k(s) := \langle T'(s), N(s) \rangle}$$

curvature of α (at s)

Remark: 1) $k : I \to \mathbb{R}$ is a smooth function

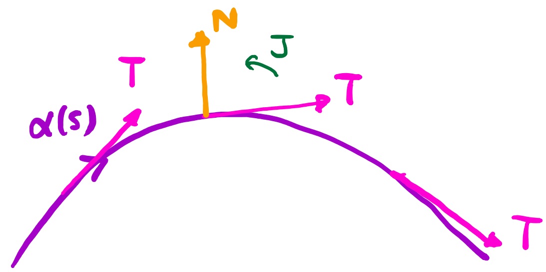
2) k has a sign:

$\boxed{k > 0}$



"T turning towards N"

$\boxed{k < 0}$



"T turning away from N"

Caution: No such interpretation for space curves!

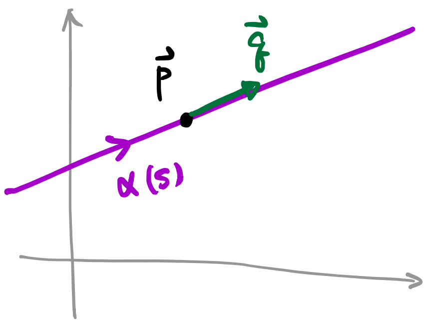
3) If $\alpha : I \to \mathbb{R}^2$ is NOT p.b.a.l. (but regular)

then "reparametrize" by $\beta = \alpha \circ \phi : J \to \mathbb{R}^2$ p.b.a.l.

define
$$\boxed{k_\alpha(t) := k_\beta(\phi^{-1}(t))}$$

$\Rightarrow$ **Note:** Curvature is invariant under reparametrization!

Example 1 : Straight line



$\boxed{k \equiv 0}$

$$\alpha(s) = \vec{p} + s \cdot \vec{q} \quad , \quad s \in \mathbb{R}$$

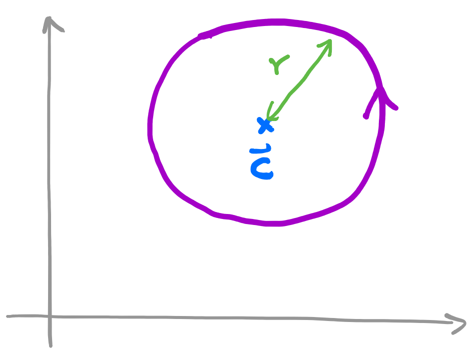
α is p.b.a.l. $\Longleftrightarrow$ $|\vec{q}| = 1$.

Frenet frame:

$$\begin{cases} T(s) = \alpha'(s) = \vec{q} & \text{constant!} \\ N(s) = J\,T(s) = J\vec{q} \end{cases}$$

$$k(s) = \langle T'(s), N(s) \rangle \equiv 0 .$$

Example 2 : Circles



$\boxed{k = \frac{1}{r}}$

"Bigger circles have smaller curvature"

$$\alpha(s) = \vec{c} + r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right)$$

p.b.a.l.

Frenet frame:

$$\begin{cases} T(s) = \alpha'(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right) \\ N(s) = J\,T(s) = \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right) \end{cases}$$

Curvature:

$$\begin{aligned} k(s) &= \langle T'(s), N(s) \rangle \\ &= \left\langle \frac{1}{r}\left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right), \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right) \right\rangle \\ &= \frac{1}{r} \qquad \text{constant!} \end{aligned}$$

Exercise: Repeat the above calculation using a clockwise parametrization.

Proposition: Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l.
If $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a rigid motion, then

$$\beta = \varphi \circ \alpha : I \to \mathbb{R}^2 \quad \text{is also p.b.a.l.}$$

and

$$k_\beta(s) = \begin{cases} k_\alpha(s) & \text{if } \varphi \text{ is orientation-preserving} \\ -k_\alpha(s) & \text{if } \varphi \text{ is orientation-reversing} \end{cases}$$

Proof: Exercise!

"Curvature is a geometric quantity"

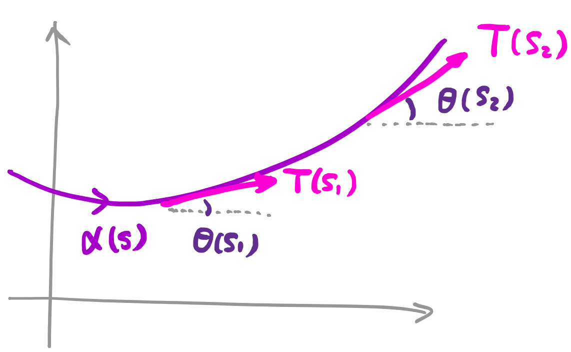
(not necessarily p.b.a.l.)

Exercise: If $\alpha : I \to \mathbb{R}^2$ is a regular plane curve, show that

$$\boxed{k_\alpha(t) = \frac{\det(\alpha'(t), \alpha''(t))}{|\alpha'(t)|^3}}$$

Exercise: Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l.

Denote $\theta(s)$ = angle of $T(s)$ measured from x-axis.



Then: $\boxed{\theta'(s) = k(s)}$

"Curvature measures the rate of turning of the unit tangent vector"